Connectivity Properties of a Packet Radio Network Model

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Abstract —A model of a packet radio network in which transmitters with range R are distributed according to a two-dimensional Poisson point process with density D is examined. To ensure network connectivity, it is shown that $\pi R^2 D$, the expected number of nearest neighbors of a transmitter, must grow logarithmically with the area of the network. For an infinite area there exists an infinite connected component with nonzero probability if $\pi R^2 D > N_0$, for some critical value N_0. We show that $2.195 < N_0 < 10.526$.

I. Introduction

A SIMPLISTIC though widely used model of a mobile packet radio network is based on the following assumptions (refer to Fig. 1).

1) Nodes in the network lie in a bounded figure of area A. We shall assume the figure to be a square.
2) *The Homogeneous Poisson Assumption:* An elemental area ds contains at most one node. The probability of this node's existence is Dds, where D is the density of nodes in the plane.
3) Each node is able to communicate with any other node that is at most R units distant from it.
4) $\pi R^2 \ll A$.
5) All nodes generate Poisson streams of traffic at an identical rate.

Under these assumptions, with the further restriction that the medium access protocol be slotted Aloha, a number of authors [3], [6], [11] have shown that for the throughput to be maximized, we must have $\pi R^2 D \sim 6$, leading to the widely held belief that six is a "magic number." In this paper it will be shown that if $\pi R^2 D$ is a fixed constant, then for sufficiently large A the network will almost surely be disconnected, implying that no magic number can exist. This, however, does not render the notion of a magic number useless. A large square is well approximated by the infinite XY plane, and Gilbert [1] has shown that there is a critical number N_0 such that if $\pi R^2 D > N_0$, the random plane network contains an infinite connected component with nonzero probability. He found N_0 to be bounded between 1.64 and 17.9. More recently, Kirkwood and Wayne [5] and Hall [2] have shown that $2.186 < N_0 < 10.588$. We shall tighten these bounds to $2.195 < N_0 < 10.526$. Returning to the finite area A, we would expect the vast majority of nodes to be contained in a single giant component if $\pi R^2 D > N_0$. Gilbert ran simulations to verify this, and found it to be true in practice. In addition, he estimated the true value of N_0 to be about 3.2.

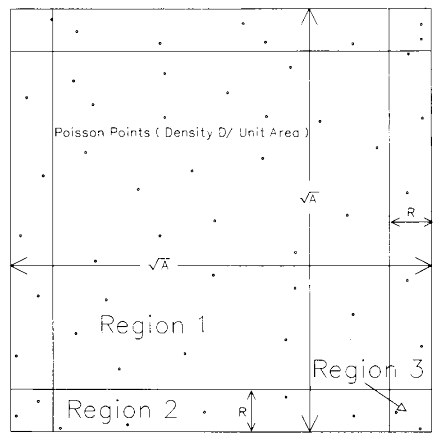

Fig. 1. Model.

The paper is organized as follows. In Section II we determine the necessary and sufficient conditions for the plane to be covered (i.e., for every point in the square to lie at a distance of R or less from some Poisson point). In Section III we present a necessary condition for the Poisson points to be connected. In Section IV Gilbert's problem [1] is re-examined, and its relation to the packet radio model is explored.

II. Covering the Plane

First, we consider the problem of covering an area with randomly located circles. Consider a square of area A in which points are generated by a two-dimensional Poisson point process of density D points per unit area. Each Poisson point is assumed to cover all points lying within a radius R of it. The question posed is the following: given a functional form for R that may depend on D and A, find $\lim_{A\to\infty} \Pr[\text{square is covered}]$.

Manuscript received November 6, 1987; revised November 1988. This work was supported in part by the Center for Advanced Technology in Telecommunications, Polytechnic University, Brooklyn, NY. This work was presented at IEEE Milcom '88, San Diego, CA, October 1988.

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IEEE Log Number 8930432.

0018-9448/89/0900-1044$01.00

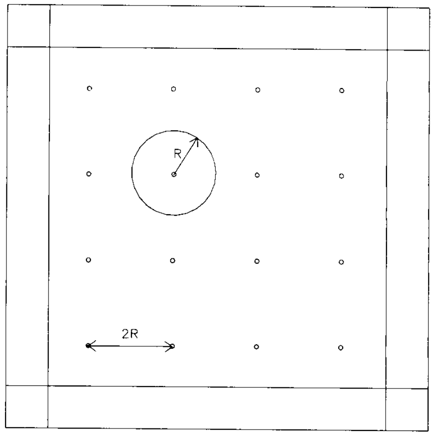

Fig. 2. Covering the plane: square grid.

If we consider the Poisson points to be base stations or repeaters in a packet radio network, the circle of radius R would define the transmitting or receiving range of the station. Clearly, if the square is covered, every point in it will have access to at least one station.

Theorem 1: For any $\epsilon > 0$, if $R = \sqrt{(1-\epsilon)\ln A/\pi D}$, then $\lim_{A\to\infty} \Pr[\text{square covered}] = 0$.

Proof: On the square of area A, construct a square lattice of side $2R$ containing $\lfloor(\sqrt{A}/2R)\rfloor^2$ points as shown in Fig. 2. The lattice is drawn so that it is centered over region 1. Then if $\sqrt{A}$ is a multiple of $2R$, its outermost rows will lie on the boundary between regions 1 and 2, and if not, the lattice points will be contained entirely within region 1. If the plane is to be covered, every lattice point must be covered. For a lattice point to be covered, a Poisson point must lie within a circle of radius R centered at the lattice point. Therefore, in region 1 (refer to Fig. 2) we can write

$$\Pr[\text{a specified lattice point is covered}] = 1 - e^{-\pi R^2 D}.$$

Let Y be a random variable that counts the number of uncovered lattice points. Then

$$\begin{aligned}\Pr[Y=0] &= \left(1 - e^{-\pi R^2 D}\right)^{\lfloor(\sqrt{A}/2R)\rfloor^2}\\ &= (1+o(1))\exp\left(-\frac{A^\epsilon}{4R^2}\right)\\ &= o(1).\end{aligned} \tag{1}$$

It follows that in the limit as $A \to \infty$, the square is almost surely not covered.

Theorem 2: For any $\epsilon > 0$, if

$$R = \sqrt{\frac{(1+\epsilon)\ln A}{\pi D}},$$

then $\lim_{A\to\infty} \Pr[\text{squared covered}] = 1$.

Proof: Fix $\epsilon < 0.5$, and tile the plane with squares of side $\epsilon R/4\sqrt{2}$. The tiles adjacent to the boundaries may be smaller, though this is inconsequential. Clearly, the plane is covered if and only if each of these tiles is covered. For a tile to be covered it is sufficient that one or more Poisson points lie in a circle of radius $(1-(\epsilon/4))R$ centered at the center of each tile. The expectation of the number of uncovered tiles is easily shown to be

$$O\left(A^{-(\epsilon/4)+(\epsilon^2/2)}\right). \tag{2}$$

The predominant contribution to the expectation comes from Region 2. For $\epsilon < 0.5$, this quantity is $o(1)$, and as the probability that a nonnegative integer random variable exceeds 0 is upper-bounded by its expectation [8, pp. 10–11], the probability that one or more tiles is uncovered $\to 0$ as $A \to \infty$. The truth of the theorem follows from the observation that the probability of the plane being covered is an increasing function of ϵ.

Observation: To guarantee that the area is covered, a node must have $\pi[(1+\epsilon)\ln A/\pi D]D$ or a little more than $\ln A$ nearest neighbors (Poisson points that lie at a distance of R or less from it) on the average.

III. Connectivity

Once again consider our square of area A in which points or repeaters are generated by a two-dimensional Poisson point process of density D. Each point is assumed to be connected to all points at a distance of R or less from it. If the Poisson points represent repeaters, and if the network is connected, a transmitter that lies within the range of a repeater can communicate with a receiver located within the range of any other repeater, and this points to a simple way to ensure radio coverage of a region. If, on the other hand, the Poisson points are thought of as transceivers, as in a mobile radio network, the connectedness of the network allows communication between any pair of transceivers. Once again, given a functional form for R that may depend on D and A, we investigate the behavior of $\lim_{A\to\infty} \Pr[\text{network is connected}]$.

Though connectivity and coverage are independent—neither implies the other—intuitively one would expect the two properties to have very similar thresholds. A simple argument to justify this assertion goes as follows. Suppose first that $R = \sqrt{(1-\epsilon)\ln A/\pi D}$. Then the expectation of the uncovered area is $O(A^\epsilon)$ which $\to \infty$ as $A \to \infty$, and one would expect an uncovered patch to separate two components in the graph. Likewise, if $R = \sqrt{(1+\epsilon)\ln A/\pi D}$, the square is almost surely covered, implying that the Poisson points lie very "close" to each other. We therefore expect the probability of the network being connected to be very high. This heuristic argument is put on a firmer footing in the next theorem.

Theorem 3: For any $\epsilon > 0$, if $R = \sqrt{(1-\epsilon)\ln A/\pi D}$, then $\lim_{A\to\infty} \Pr[\text{network is connected}] = 0$.

Proof: Referring to Fig. 1, if a single node that lies in region 1 is isolated, the network is not connected. To show that an isolated node exists with high probability, we proceed as follows.

1) Find the first two moments of the number of isolated nodes.
2) Use Chebyshev's inequality to show that the probability of finding an isolated node $\to 1$ as $A \to \infty$.

Let X be a random variable that counts the number of isolated nodes in region 1. Then

$$\Pr[\text{a specified node is isolated}] = e^{-\pi R^2 D}$$

and

$$E[X] = (\sqrt{A} - 2R)^2 D e^{-\pi R^2 D}. \tag{3}$$

If $R = \sqrt{(1-\epsilon)\ln A/\pi D}$, we have

$$E[X] = (\sqrt{A} - 2R)^2 D e^{-(1-\epsilon)\ln A} = (1 - o(1)) D A^{\epsilon}. \tag{4}$$

The expected number of isolated nodes grows without bound as A is increased. To find the second moment, define indicator random variables $\{x_i\}$, $i \geq 1$, such that $x_i = 1$ if the ith node is isolated, and 0 if it is not. Then $X = \sum_i x_i$, and

$$E[X^2] = E\left[\sum_i x_i^2 + \sum_{i \neq j} x_i x_j\right] = E[X] + (1 + o(1)) E^2[X], \tag{5}$$

provided that $\pi R^2 D > (1+\epsilon)\ln A$.

In the last step we use the fact that $x_i = x_i^2$ to conclude that the first term is $E[X]$. To see that the second term is $(1 + o(1))E^2[X]$, first evaluate it conditioned on the existence of exactly m points, and then remove the condition using the fact that the number of nodes in region 1 is a Poisson random variable. Finally, use the condition on $\pi R^2 D$ to conclude that the subdominant terms are $o(1)$ with respect to the dominant term. Using a variation of Chebyshev's inequality [8, p. 138], we get

$$\Pr[X = 0] \leq \frac{E[X^2] - E^2[X]}{E^2[X]} = \frac{1}{E[X]} + o(1) = o(1).$$

It follows that the probability of the graph being connected also goes to 0 as $A \to \infty$.

Corollary: There can be no magic number, as the expected number of nearest neighbors needed to ensure connectivity grows logarithmically with the area of the square in which the points lie.

We conjecture, but cannot prove, that if

$$R = \sqrt{(1+\epsilon)\ln A/\pi D}, \quad \epsilon > 0,$$

the graph is almost surely connected. The conjecture is based on the observation that if $R = \sqrt{(1+\epsilon)\ln A/\pi D}$ the square is almost surely covered, implying that the Poisson points lie very close to each other.

IV. The Infinite Component

As indicated in the corollary to Theorem 3, no magic number can exist. In this section we explore the properties of random plane networks of low density and use them to conclude that the concept of a magic number is useful in practice. Gilbert [1] showed that if the Poisson process was assumed to generate points over the entire XY plane and if the average number of nearest neighbors of a point exceeded some fixed constant N_0, an infinite component would exist with nonzero probability. He found $1.64 < N_0 < 17.9$. (Due to a typographical error, the lower bound on N_0 in [1] is given as 1.75.) More recently, Kirkwood and Wayne [5] and Hall [2] have shown that $2.186 < N_0 < 10.588$. Note that the existence of an infinite component does *not* imply that all the nodes are connected. There will, by Theorem 3, be infinitely many isolated nodes. When the area of the square is finite, no infinite component can exist, though we might reasonably expect most of the nodes to belong to a giant component. Gilbert carried out simulations and found this to be true in practice. From the simulation, he concluded that $N_0 \sim 3.2$. It has been conjectured that the true value of N_0 is π, though no proof of this has been found. The following theorem due to Robbins [10] now proves useful.

Theorem 4: Let S be a random Lebesgue measurable subset of R_n with measure $\mu(S)$. For any point $x \in R_n$, let $p(x) = \Pr[x \in S]$. Define

$$g(x,S) = \begin{cases} 1, & \text{if } x \in S \\ 0, & \text{otherwise.} \end{cases}$$

Then assuming that the function $g(x,S)$ is a measurable function of the pair (x,S), the expectation of the measure of S is given by the Lebesgue integral of the function $p(x)$ over R_n.

Consider a random plane network generated so that the average number of nearest neighbors of a randomly chosen node is six. By Robbins theorem, the fraction of the square that is uncovered is approximately e^{-6}, and the fraction that is singly covered is about $6e^{-6}$. Together, they account for less than 1.75 percent of the area of the square. It is reasonable to expect much of the singly covered area to be composed of isolated nodes and portions of the boundary of the giant component (assuming that $N_0 < 6$). If the portion of the singly covered area that consists of isolated nodes is erased, the remaining portion should not differ appreciably from our model of a two-dimensional Poisson point process of density D points/unit area. In addition, as the uncovered and singly covered area is a small fraction of the total area, the giant component would tend to cover most of the plane. These facts taken together explain the generally good agreement between theory and simulation reported in [3].

Finally we present, without proof, two theorems that provide tighter bounds on N_0 than those found in [2]. To lower bound N_0 we relate this problem to a multiclass $M/D/1$ queue. Instability in the queueing system corresponds to the existence of an infinite component. The upper bound is proved by considering a percolation process on a triangular lattice. The existence of an infinite chain in the lattice implies the existence of an infinite component in the plane. The interested reader is referred to [9].

Theorem 5: $N_0 > 2.195$.

Theorem 6: $N_0 < 10.526$.

V. Conclusion

A model of a packet radio network has been examined, and it has been shown that no optimal number of nearest neighbors or "magic number" can exist. The notion of a magic number has been shown to be useful, however, and an explanation for the generally good agreement between theory and simulation has been presented. A number of open questions remain. The single most important one concerns the homogeneity assumption made about the Poisson process. Extending these results to the inhomogeneous case will mark an important step forward. Another assumption that bears investigation is that of each node being connected to all nodes that lie within a circle of radius R around it. Certainly, this is not valid in many environments of interest. These questions are significantly more difficult to resolve than those examined in this paper. We do, however, conjecture that the following hold.

Conjecture 1: The conditions for connectivity and coverage are insensitive to the shape of the region that a Poisson point covers. This may not be unduly difficult for convex figures (such as the circle considered in this paper).

Conjecture 2: If the Poisson process is nonhomogeneous, then the "radius of influence" around each node that is needed to guarantee almost sure connectivity and almost sure coverage will be such that its expected number of nearest neighbors is $\geq (1+\epsilon)\ln A$.

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