On the Throughput of Degenerate Intersection and First-Come First-Served Collision Resolution Algorithms

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Abstract—It is shown that, for a ternary feedback random access channel with a Poisson arrival process, 0.5 is an upper bound to the throughput for all "degenerate intersection" algorithms (DIA's) and first-come first-served algorithms (FCFSA's). As a by-product, the nested FCFSA with the largest throughput is found for the random access channel with a Bernoulli arrival process with parameter p. For $p \geq 0.018$, this algorithm has the highest throughput over all DIA's and FCFSA's. Lastly, it is shown that, for some values of p, a non-DIA, non-FCFSA has a higher throughput than the optimum DIA or FCFSA.

I. Introduction

We consider the problem of a large distributed population of occasionally active users attempting to communicate over a single shared channel such as a satellite link or a coaxial cable. This problem is referred to as the multiple-access problem. Whenever two or more users attempt to transmit messages at the same time, a collision occurs and the messages involved are lost. When this happens a collision resolution algorithm (CRA) is used to control retransmissions of the lost messages so that eventually they are successfully transmitted. The focus of much research has been to devise collision resolution algorithms that maximize the channel throughput while keeping the expected message delay finite. Finding an upper bound on the throughput achievable by CRA's has also been of considerable interest to workers in this area.

The slotted ALOHA system [1] included a primitive form of a collision resolution algorithm. A tree search algorithm devised by Capetanakis [2] and later improved by Massey [3] was the first algorithm that guaranteed finite message delays for a range of offered traffic rates. Gallager developed a time-window algorithm [4] which transmits messages in a first-come first-served (FCFS) order and achieves a maximum throughput (in packets per slot) of 0.4871. Humblet and Mosely [5] refined this algorithm, while still maintaining the FCFS property, and obtained the highest throughput (≈ 0.4877) achieved thus far for any FCFS collision resolution algorithm. Recently, Vvedenskaya and Pinsker [6] have introduced a modification of the Humblet–Mosely algorithm, leading to an improvement in the throughput in the seventh decimal place. Interestingly, this modified algorithm is not FCFS, but belongs to a different class of algorithms called degenerate intersection algorithms (DIA's), to be defined later. To this date, the throughput achieved in [6] is the best lower bound to the maximum throughput of all algorithms.

The first upper bound to the maximum throughput was obtained by Pippenger [7] who, by using an information-theoretic approach, obtained the upper bound 0.744. Humblet [8] decreased this bound to 0.704. Molle [9] was the first to introduce the concept of a helpful genie. He showed that the optimal genie-aided algorithm achieved a throughput of 0.6731. Since any algorithm without a genie could not exceed the throughput achieved by an optimal genie-aided algorithm, this number constitutes an upper bound to the throughput for all algorithms. Cruz and Hajek [10] used a less helpful genie to obtain the tighter upper bound of 0.6215. The tightest upper bound to the maximum throughput obtainable over all algorithms to date is 0.587 and is due to Tsybakov and Mikhailov [11]. Molle [12] used a genie-aid algorithm to obtain the tighter bound of 0.508 for the restricted class of degenerate intersection algorithms (DIA's). A still tighter bound of 0.5 has been obtained by Cruz [13] for FCFS algorithms (FCFSA's). The main result of this paper is to tighten the bound on DIA's to 0.5. The above results have been computed for the case when message arrivals are Poisson. Another model we will be considering in this paper is the Bernoulli arrival model studied extensively by Molle [9], [12].

In Section II of this paper, we define the system model and the classes of collision resolution algorithms of interest in this paper. In Section III, we obtain the optimal nested FCFSA for the Bernoulli arrival model. We show that the throughput of DIA's cannot exceed 0.5 for the Poisson arrival model in Section IV. Using a method different from that used in [13], we show that 0.5 is also an upper bound on the throughput achievable by FCFSA's. In Section V, we consider a mixing algorithm which is neither a DIA nor a FCFSA, but which achieves a throughput marginally higher than the optimal DIA. Finally, in Section VI we present our conclusions.

II. The Model and Some Definitions

In this section we provide a formal description of the multiple-access system considered in this paper and define

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various classes of collision resolution algorithms. Consider a large and independent set of users sharing a common channel. The users generate messages of fixed duration called packets. The channel is slotted and the duration of each slot equals the length of a packet. Transmissions of packets are timed to coincide with these slots. At the end of each slot, every user is informed whether the slot was idle (contained no packets), had a successful transmission (contained one packet) or experienced a collision (contained two or more packets). Throughput is defined as the expected number of successful transmissions per slot. The message arrival time axis is discrete and consists of a countably infinite number of arrival points. In the Bernoulli arrival model, each arrival point has a probability p of containing one message and $(1-p)$ of containing no message. The message arrivals at different arrival points are independent of each other. The message arrival process thus corresponds to an infinite sequence of independent Bernoulli trials, each with parameter p. If there are N such arrival points per unit of time and if we let $N \to \infty$ and $p \to 0$ in such a way that the product $S = Np$ is kept constant, then the Bernoulli process will tend to the Poisson process with parameter S and a continuous arrival time axis.

A collision resolution algorithm can be viewed as a sequence of tests in each of which a set of arrival points are enabled, i.e., only message arrivals in that set are transmitted in the next slot. The composition of the set of arrival points to be enabled at any point in time is determined from the entire past history of the composition of previously enabled sets and their corresponding test results. We shall be concerned with two classes of collision resolution algorithms, DIA's and FCFSA's. A collision resolution algorithm is considered to be optimal over a class of algorithms if it achieves the highest throughput possible for that class of algorithms.

The following definition of DIA's is due to Molle [12]. Let S be a set of arrival points. Define $\theta_t(S)$ to be the smallest number of messages that could be contained in S, given the complete channel history, up to the start of the tth slot. Assume that, at the start of the tth slot, there are exactly n_t distinct subsets (not necessarily disjoint), $B_1^t, B_2^t, \cdots, B_{n_t}^t$, for which $\theta_t(B_i^t) > 0$. Such subsets will be called busy subsets. Let U_t be the set of points about which nothing is known beyond the *a priori* information at the tth slot. Let K_t be the set of points for which it is known that all contained message packets have been successfully transmitted by the tth slot. Let E_t be the enabled set at the tth slot.

Definition: Degenerate intersection algorithms (DIA's) are defined by the following constraint on E_t

$$E_t \cap B_i^t \in \{E_t, B_i^t, \varnothing\}$$

for all $i = 1, 2 \cdots, n_t$, $t > 0$ where $n_1 = 0$.

Definition: Nested DIA's (which are a subset of the class of DIA's) are defined by a more restrictive constraint on E_t,

$$E_t \cap B_i^t \in \{E_t, \varnothing\}$$

for all $i = 1, 2 \cdots, n_t$, $t > 0$ where $n_1 = 0$.

Definition: First-come first-served algorithms (FCFSA's) are those algorithms which ensure that packets are (successfully) transmitted in the same order as they arrived for transmission.

Definition: Nested FCFSA's, a subset of the class of FCFSA's, is defined by the following constraints on E_t.

1) If E_t consists of k arrival points, then these k arrival points will be the k earliest arrival points which are not elements of K_t.
2) $n_t = 0$ or 1.
3) If $n_t = 0$, $E_t \cap U_t = E_t$.
4) If $n_t = 1$, $E_t \cap B_1^t = E_t$.

DIA's are a class of algorithms which include, in addition to some FCFSA's (such as the Gallager and Humblet–Mosely algorithms), coin-tossing algorithms such as the Capetanakis and Massey algorithms. The Gallager algorithm is also an example of a nested FCFSA. It is clear from the definitions that nested FCFSA's form a subset of nested DIA's. However, the class of DIA's does not include *all* possible FCFSA's.

III. The Optimal Nested FCFSA

We now outline a method for finding the optimal nested FCFSA. The method is similar to that employed by Humblet and Mosely [5] except that we use the Bernoulli arrival model instead of the Poisson arrival model. The algorithm is modeled as a Markov process with three classes of states. The result of enabling a set of arrival points yields a state transition. Our aim is to maximize the expected number of successes per state transition. This is equivalent to maximizing the expected number of successes per slot or the throughput. Two results from Markovian decision theory, the value iteration algorithm [14] and the Odoni bound [15], are used to compute the optimal algorithm and its resultant throughput. These are described next.

Consider a finite state, discrete-time, ergodic Markov system. After each transition, the system is in one of N states $i = 1, 2, \cdots, N$. For each state i, an action $k = 1, 2, \cdots, K_i$ is chosen. Then p_{ij}^k is the probability of transition to state j if the process is in state i and action k was chosen. Associated with each transition from i to j, under action k, is a reward r_{ij}^k. Let the value function $v_i(n)$ be the total expected reward from the next n transitions, if the system is now in state i, and if an optimal policy is followed. The expected value can be written as

$$v_i(n+1) = \max_{1 \le k \le K_i} \left[\sum_{j=1}^{N} p_{ij}^k \left(r_{ij}^k + v_j(n) \right) \right], \quad i = 1, 2, \cdots, N. \tag{3.1}$$

Let g be the maximal expected gain per transition. The principal goal in Markov decision processes is to compute

g and to find the policy that achieves g. This can be done using the following theorem due to Odoni [15].

Theorem: Let all stationary policies have transition-probability matrices representing single-chain aperiodic Markovian processes. Define $x_i(n)$ by

$$x_i(n) = v_i(n+1) - v_i(n).$$

Then for any choice of boundary conditions $v_i(0)$

1) $x_i(n) \to g$, $1 \le i \le N$.
2) $L''(n) = \max_i x_i(n)$ is monotonically decreasing in n to g.
3) $L'(n) = \min_i x_i(n)$ is monotonically increasing in n to g.
4) Any policy A achieving the N maxima in (3.1) for all n greater than or equal to some n_0 has the maximal gain per transition.

The value iteration algorithm proceeds as follows. The value functions $v_i(n)$, $n = 1, 2, \cdots$ are computed using equation (3.1). This computation is carried out until $L''(n) - L'(n) < \epsilon$, where ϵ is the maximum allowable error in the computation of g. Then $g \approx 1/2[L''(n) + L'(n)]$, since $L'(n) \le g \le L''(n)$ from the theorem. The optimal policy which achieves the maximal gain g may be determined as the one which achieves the N maxima in (3.1) for all $n \ge n_0$.

The value iteration equations for determining the optimal nested FCFSA can be described in terms of a state space with three types of states.

H: The initial state in which $n_t = 0$. At this point all the packets yet to be transmitted are Bernoulli distributed. An *epoch* corresponds to the time between visits to state H.

$S1[L]$: The state in which $\theta_t(B_1^t) = 1$ and $|B_1^t| = L$.

$S2[M]$: This is the state in which $\theta_t(B_1^t) = 2$ and $|B_1^t| = M$.

The reward is set to 1 for a successful transmission and to 0 otherwise. As a result, the gain per state transition corresponds to the throughput. The value functions corresponding to the three types of states for the nth transition are $VH[n]$, $VS1[n, L]$ and $VS2[n, M]$. The set of value functions $\{VH[n],\ VS1[n, L]$ for $L \ge 0,\ VS2[n, M]$ for $M \ge 1\}$ is equivalent to the set of value functions $\{v_i(n);\ 1 \le i \le N\}$ defined earlier. Parameters k, a and b correspond to the number of arrival points to be enabled at the next transition. Let $q(m)$ denote the number of packets in a set of m arrival points. We then have the following relations:

$$\begin{aligned} VS2[n+1, M] = \max_{0<b<M} \{ & P[q(b)=0|q(M)\ge 2]\,VS2[n, M-b] \\ & + P[q(b)=1|q(M)\ge 2](1+VS1[n, M-b]) \\ & + P[q(b)\ge 2|q(M)\ge 2]\,VS2[n, b]\} \end{aligned} \quad (3.4)$$

with initial values of $VH[0] = 0$; $VS1[0, L] = 0$, $L \ge 0$; and $VS2[0, M] = 0$, $M \ge 1$.

Table I summarizes the results of carrying out this computation for $\epsilon = 10^{-5}$. For values of $p \ge 0.375$, the algorithm yields numerical results equal to the lower bound obtained by Molle [6]. The optimal nested FCFSA achieves throughputs higher than 0.5 for $p \ge 0.018$ (see Fig. 1). For values of $p \ge 0.123$, the optimal nested algorithm was found to correspond to a "halving algorithm." A "halving algorithm" is a nested FCFSA, similar to Gallager's algorithm [4], such that whenever $\theta_t(B_1^t) = 2$, $|B_1^t| = M$,

TABLE I
THROUGHPUT OF THE OPTIMAL DIA (FCFSA) AND THE MIXING ALGORITHM

p	Optimal DIA (FCFSA)	Mixing	DIA Upper Bound (Molle)
0.05	0.51459	0.51467[a]	0.53051
0.10	0.53322	0.53359[a]	0.54676
0.20	0.56389	0.56476[a]	0.57550
0.30	0.60116	0.60116[b]	0.60534
0.40	0.62241	0.62241[b]	0.62241

[a] Mix1 incorporated.
[b] Mixing not employed.

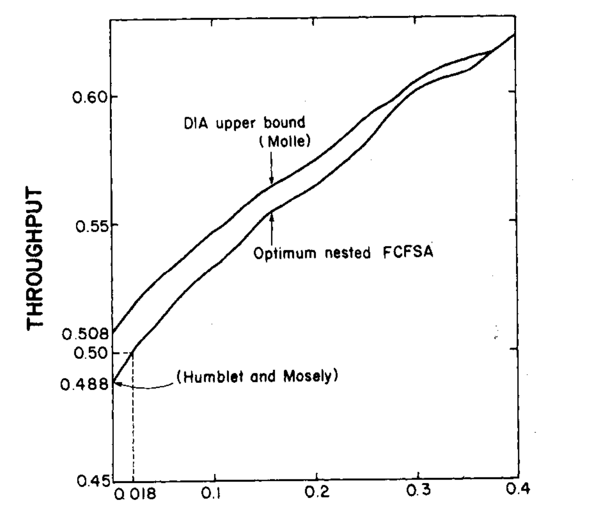

Fig. 1. Throughput versus Bernoulli probability p.

$$\begin{aligned} VH[n+1] = \max_{k>0} \{ & P[q(k)=1] + \{P[q(k)=1] + P[q(k)=0]\}\,VH[n] \\ & + P[q(k)\ge 2]\,VS2[n, k]\} \end{aligned} \quad (3.2)$$

$$\begin{aligned} VS1[n+1, L] = \max_{0<a\le L} \{ & P[q(a)=0|q(L)\ge 1]\,VS1[n, L-a] \\ & + P[q(a)=1|q(L)\ge 1](1+VH[n]) \\ & + P[q(a)\ge 2|q(L)\ge 1]\,VS2[n, a]\} \end{aligned} \quad (3.3)$$

$\lfloor M/2 \rfloor$ of the points from B_1^t are enabled in the next slot ($\lfloor \cdot \rfloor$ is the floor function), whereas if $\theta_t(B_1^t) = 1$ then all of the points in B_1^t are enabled in the next slot. For values of $p \leq 0.123$, the optimal algorithm occasionally enables less than half of a collision set, and thus differs from the halving algorithm.

IV. The 0.5 Bound for DIA's and FCFSA's

In this section we show that the upper bound on the throughput of all DIA's and FCFSA's is 0.5 for the Poisson arrival model. In addition, the optimal nested FCFSA will be shown to be identical to both the optimal FCFSA and the optimal DIA for $p \geq 0.018$. An outline of the method used to obtain the throughput bound follows. We first show (Lemmas 1, 2, and 3) that the optimal FCFSA and the optimal DIA are identical to the optimal nested FCFSA provided the throughput achieved by the latter exceeds 0.5. Because we showed in the last section that the optimal nested FCFSA achieves throughputs higher than 0.5 for values of $p \geq 0.018$, the optimal nested FCFSA must also be the optimal DIA (FCFSA) for all $p \geq 0.018$. We further show in Lemma 4 that the throughput of the optimal DIA increases monotonically with p. Consequently we show in Theorem 1 that, since the optimal DIA achieves a throughput of 0.5 for $p \approx 0.018$, the throughput of the optimal DIA cannot exceed 0.5 in the Poisson limit as $p \to 0$. In Theorem 2, we show that the upper bound on the throughput of FCFSA's is also 0.5.

Lemma 1: For either the Bernoulli arrival model with parameter p or the Poisson arrival model, if the throughput of the optimal nested FCFSA exceeds 0.5, it will also be optimal over the entire class of FCFSA's.

Proof: The method we will use here is similar to that used by Molle [12]. The proof that follows is for the case of the Bernoulli arrival model. The proof for the case of the Poisson arrival model is similar and is therefore not included. We will prove this lemma by contradiction. That is, we will show that FCFSA's which have throughput greater than 0.5 and are not nested FCFSA's cannot be optimal. A nonnested FCFS strategy can enable a strict superset of a busy set. We now show, using a genie argument, that an optimal FCFSA will never enable a superset of a busy set, if the throughput of the optimal FCFSA exceeds 0.5.

Let us consider a genie-aided FCFSA which possibly enables a superset of a busy set, and compare it with an unaided algorithm which never enables a superset of a busy set but is otherwise identical. Whenever the genie-aided FCFSA enables E_t, a strict superset of a busy set B_1^t, the genie labels the first arrival, if any, in the set $E_t - B_1^t$. The genie also labels all idle points before the first arrival in $E_t - B_1^t$. If enabling E_t leads to a success, it is clear that any information the genie gives is redundant. An idle is impossible since B_1^t is a busy set. If there is a collision *and* the genie labels an arrival, the successful transmission of this arrival requires two slots, the slot corresponding to the collision which led to the labeling of the arrival and the future slot when it is successfully transmitted. Tag the two slots required to transmit the genie-labeled arrival. Since $\theta_t(E_t) = 2$ and $E_t - B_1^t$ had at least one arrival, we get no new information about B_1^t. If there is a collision and all the arrival points in $E_t - B_1^t$ are labeled idle by the genie, we can surmise that $\theta_t(B_1^t) = 2$. But an unaided algorithm would have gained that information by enabling B_1^t alone. Thus, if one excludes the tagged slots, the unaided nested FCFSA would perform in exactly the same manner as the genie aided nonnested FCFSA. The ratio of the number of genie-labeled arrival transmissions to the (tagged) slots required to identify and transmit them is 0.5. Therefore, if the unaided, nested algorithm achieves a throughput higher than 0.5, the genie-aided nonnested algorithm can only have a lower throughput than the nested algorithm. Hence any non-nested FCFSA will have a lower throughput than the optimal nested FCFSA provided the throughput of the latter exceeds 0.5, which proves the lemma.

The proof of the following lemma is given by Molle [12].

Lemma 2: If the throughput of the optimal nested DIA exceeds 0.5, it will also be optimal over the entire class of DIA's for the Bernoulli arrival model with parameter p.

Lemma 3: For the nested DIA, for all $t > 0$,

1) $\theta_t(B_i^t) = 1$ or 2 for all i and
2) $B_i^t \cap B_j^t = \varnothing$ for all $i \neq j$.

Proof: This result can be shown by induction on t. Let us define $B^t = \{B_i^t;\ 1 \leq i \leq n_t\}$. For $t = 1$, $B^t = \varnothing$ and therefore the result is true. Assume it is true for $t = T$ and consider all the permissible cases for a nested DIA for $t = T + 1$.

Case 1: $E_T \cap B_i^T = \varnothing$ for all $B_i^T \in B^T$ (i.e., $E_T \cap U_T = E_T$).

If enabling E_T results in an idle or a success, then $B^{T+1} = B^T$. If enabling E_T results in a collision, then $B^{T+1} = B^T \cup \{E_T\}$ with $\theta_{T+1}(E_T) = 2$.

Case 2: $E_T \cap B_i^T = E_T$ (i.e., E_T is a subset of B_i^T) and $E_T \cap B_j^T = \varnothing$ for all $j \neq i$.

If enabling E_T results in an idle, then $B^{T+1} = (B^T - \{B_i^T\}) \cup \{B_i^T - E_T\}$ and $\theta_{T+1}(B_i^T - E_T) = \theta_T(B_i^T) = 1$ or 2. If enabling E_T leads to a success, then $B^{T+1} = B^T - \{B_i^T\}$ if $\theta_T(B_i^T) = 1$. If $\theta_T(B_i^T) = 2$, $B^{T+1} = (B^T - \{B_i^T\}) \cup \{B_i^T - E_T\}$ with $\theta_{T+1}(B_i^T - E_T) = 1$. If there is a collision, then $B^{T+1} = (B^T - \{B_i^T\}) \cup E_T$, with $\theta_{T+1}(E_T) = 2$.

In addition, all the new busy sets created in both cases are disjoint from all the others. We have shown that the lemma is true for $t = T + 1$ if it is true for $t = T$, thus proving the lemma.

From Lemma 3 it is clear that a nested DIA either enables a subset of a busy set disjoint from all other busy sets or a subset of the Bernoulli distributed points. By the independence property of the Bernoulli arrival process, it follows that the result of enabling a subset of a busy set or U_t will be independent of enabling a subset of any other busy set. Also, to preserve the stability of a nested DIA, all busy sets have to be eventually resolved. Therefore, there is no loss of generality if we enable sets in a nested FCFS manner, i.e., resolve each busy set as it is created before enabling another subset of U_t. Therefore, since any nested DIA has an equivalent nested FCFSA, the optimal nested FCFSA obtained in the last section is also the optimal nested DIA.

Lemma 4: The throughput of the optimal DIA increases monotonically with p.

Proof: The proof is similar to that used by Molle [9] for optimal algorithms (over all classes). Consider two Bernoulli arrival processes with parameters p_1 and p_2, $p_2 > p_1$. Let the throughput achieved by the optimal DIA for the Bernoulli arrival process with parameter p_2 be $T(p_2)$. Suppose there exists a genie which identifies certain idle arrival points in the Bernoulli arrival process with parameter p_1 in the following way. The genie looks at each idle arrival point and independently tags it as an idle (with probability $p_g = (1 - p_1/p_2)/(1 - p_1)$) or ignores it (with probability $1 - p_g$). The untagged points now correspond to a Bernoulli arrival process with parameter p_2. Since there is no advantage in enabling the tagged idle points, the maximum throughput achievable by the genie-aided algorithm for a Bernoulli arrival process with parameter p_1 is that achieved by the optimal algorithm for a Bernoulli process with parameter p_2, i.e., $T(p_2)$. Therefore the throughput achieved by an optimal unaided algorithm for a Bernoulli process with parameter p_1 cannot exceed $T(p_2)$, i.e., $T(p_1) \leq T(p_2)$. Since this result is true for any pair (p_1, p_2) where $p_1 < p_2$, the lemma follows.

Theorem 1: For the Poisson arrival model, the throughput of the optimal DIA cannot exceed 0.5.

Proof: In the previous section we found the optimal nested FCFSA. We have shown, using Lemma 3, that the optimal nested FCFSA would also be the optimal nested DIA. But the optimal nested FCFSA (DIA) achieved throughputs higher than 0.5 for $p \geq 0.018$. Therefore, by Lemma 2, it is also the optimal DIA for $p \geq 0.018$. For a value of $p \approx 0.018$, the optimal DIA achieved a throughput of 0.5. Consequently, by Lemma 4, the optimal DIA cannot achieve throughputs greater than 0.5 for $p < 0.018$. In particular, this bound is valid if we let $p \to 0$ and $N \to \infty$ in such a way that the product $S = Np$ is kept constant, so that the Bernoulli process will tend to the Poisson process with parameter S. Therefore, for the Poisson arrival model, the throughput of the optimal DIA cannot exceed 0.5.

Theorem 2: For the Poisson arrival model, the throughput of the optimal FCFSA cannot exceed 0.5.

Proof: Consider the two following statements.

Statement A: For the Poisson arrival model, the optimal FCFSA achieves a throughput greater than 0.5.

Statement B: For the Poisson arrival model, there exists a DIA with a throughput greater than 0.5.

In Lemma 1 we showed that if Statement A were true, the optimal FCFSA would be a nested FCFSA. But a nested FCFSA is also a (nested) DIA. Therefore, if Statement A were true, it would imply that Statement B is also true. But we know from Theorem 1 that Statement B is false. Thus Statement A is false, which proves the theorem.

V. The Mixing Algorithm

In this section we show, by introducing a mixing algorithm, that a non-DIA, non-FCFSA can achieve higher throughputs than the optimal FCFSA (DIA) for some values of p. Consider the case of three arrival points (a_1, a_2, a_3), two or more of which are known to have message packets awaiting transmission. In the usual FCFSA, a_1 (or a_1 and a_2) would be the next point to be tested. Instead (see Fig. 2) we enable a_1 along with b_1, a point taken from the unknown (Bernoulli distributed) set. We will refer to the procedure shown in Fig. 1 as Mix1. If, instead of using the FCFS procedure for the halving algorithm, Mix1 is incorporated, improvements in throughput over the optimal FCFSA (DIA) can be realized (see Table I). A set of recursive equations can be used to compute the throughput of the halving algorithm, which can then be modified to include Mix1. The functions involved are defined below.

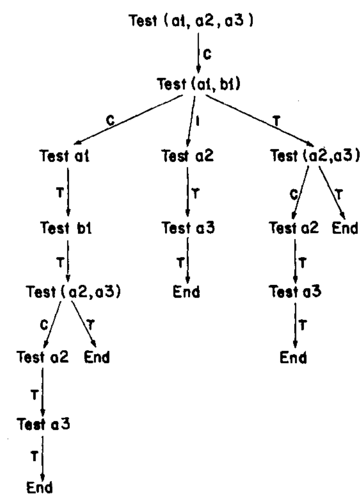

Fig. 2. Mixing with one arrival point from unknown set. *I*: Idle. *T*: Successful transmission. *C*: Collision.

$H[x]$: The expected number of queries in one epoch when x points are initially enabled.

$Gi[M]$: The expected number of queries required to resolve a set of M arrival points, i or more of which have packets to transmit, $i = 1, 2$.

$N[x]$: The expected number of arrival points resolved in one epoch when x points are initially enabled.

$Mi[L]$: The expected number of arrival points resolved when L points, with i or more packets to transmit, are processed by the algorithm, $i = 1, 2$.

It can be shown that C, the throughput of the algorithm, is

$$C = E[\text{successes/epoch}]/E[\text{queries or slots/epoch}] \quad \{\text{by definition}\}$$
$$= \max_{x \geq 2} pN[x]/H[x],$$

The recurrence equations required to compute $H[x]$ and $N[x]$ in order to carry out the above maximization are the following

$$H[x] = 1 + \{1 - (1-p)^x - xp(1-p)^{x-1}\} G2[x], \quad x > 1 \qquad (5.1)$$

$$\begin{aligned} G2[K] = 1 + \{ & P[q(y1) = 0 | q(K) \ge 2] G2[K - y1] \\ & + P[q(y1) = 1 | q(K) \ge 2] G1[K - y1] \\ & + P[q(y1) \ge 2 | q(K) \ge 2] G2[y1]\}, \\ & K \ge 3, \quad y1 = \lfloor K/2 \rfloor \end{aligned} \qquad (5.2)$$

$$G1[L] = 1 + P[q(y2) \ge 2 | q(L) \ge 1] G2[L], \quad L \ge 3, \quad y2 = L. \qquad (5.3)$$

Boundary conditions:

$$G1[1] = 1.0$$

$$G1[2] = 1.0 + 2\{1 + (1-p)^2 - 2(1-p)\} / (1 - (1-p)^2)$$

$$G2[1] = 0.0$$

$$G2[2] = 2.0;$$

$$N[x] = x(1-p)^x + x^2 p(1-p)^{x-1} + \{1 - (1-p)^x - xp(1-p)^{x-1}\} M2[x], \quad x > 1. \qquad (5.4)$$

$$\begin{aligned} M2[K] = & P[q(y1) = 0 | q(K) \ge 2]\{M2[K - y1] + y1\} \\ & + P[q(y1) = 1 | q(K) \ge 2]\{M1[K - y1] + y1\} \\ & + P[q(y1) \ge 2 | q(K) \ge 2] M[y1], \\ & K \ge 3, \quad y1 = \lfloor K/2 \rfloor \end{aligned} \qquad (5.5)$$

$$\begin{aligned} M1[L] = & P[q(y2) = 1 | q(L) \ge 1] y2 \\ & + P[q(y2) \ge 2 | q(L) \ge 1] M2[y2], \\ & L \ge 3, \quad y2 = L. \end{aligned} \qquad (5.6)$$

Boundary conditions:

$$M1[1] = 1.0$$

$$M1[2] = 2.0$$

$$M2[1] = 0.0$$

$$M2[2] = 2.0.$$

By computing the recursive equations (5.1) to (5.6) and carrying out the maximization with respect to x, we can get the optimal FCFS halving algorithm. To compute the throughput for the mixing algorithm Mix1, we simply introduce four more boundary conditions. In order to do this, we consider all the possible states $(a1, a2, a3, b1)$, count the number of tests required for each state, and compute the number of tests required to get

$$G1[3] = 1.0 + \left((3p^2(1-p) + p^3) / (1 - (1-p)^3) \right) G2[3] \qquad (5.7)$$

$$G2[3] = p^2[7(1-p^2) + 16p(1-p) + 6p^2] / [1 - (1-p)^3 - 3p(1-p)^2] \qquad (5.8)$$

$$M1[3] = 3.0 + (3p^2(1-p) + p^3) / (1 - (1-p)^3) \qquad (5.9)$$

$$M2[3] = 4.0. \qquad (5.10)$$

We now carry out the recursions for values of $K, L \ge 4$ instead of $K, L \ge 3$. It was found that the mixing algorithm leads to improvements in throughput over the optimal FCFSA (DIA) algorithm for values of $p \le 0.31$ when a situation involving a collision set of three points occurs with non-zero probability (see Table I).

VI. Conclusions

In the first part of this paper we showed that, for a Poisson arrival model, an upper bound on the throughput for all DIA's and FCFSA's is 0.5. In the second part of the paper we showed that, for some values of the parameter p for the Bernoulli arrival model, a non-DIA, non-FCFSA has a higher throughput than the optimum DIA or FCFSA.

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